

Three Principles for Quantum Gravity

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Abstract

We postulate that the fundamental principles of Quantum Gravity are diffeomorphism symmetry, unitarity, and locality. Local observables are compatible with diffeomorphism symmetry in the presence of diff anomalies, which modify the symmetry algebra upon quantization. We describe the generalization of the Virasoro extension to the diffeomorphism algebra in several dimensions, and its off-shell representations. These anomalies can not arise in QFT, because the Virasoro-like cocycles are functionals of the observer's spacetime trajectory, which is not present in QFT. Possible implications for physics are discussed.

1 The postulate

All known physical phenomena are described by two theories: General Relativity (GR), which describes gravity, and Quantum Field Theory (QFT), which describes everything else. For the past 85 years, physicists have sought to unify these two theories into a single theory of Quantum Gravity (QG). Alas, GR and QFT are mutually incompatible, and despite an immense amount of work by many leading physicists, there has been no clear progress. In particular, the origin of mass quantization (why is $m_p \approx 1836 \cdot m_e$?) remains a complete mystery.

In view of this failure, I propose to take a step back and reexamine the fundamental principles that QG should rest upon. A radical possibility is that QG simply combines the fundamental properties of GR and QFT:

Postulate 1 (Main postulate, physical version) *Quantum Gravity has the following properties:*

1. *Spacetime diffeomorphism symmetry (the gravity property).*
2. *Unitarity and energy bounded from below (the quantum property).*
3. *Locality (the field property).*

None of the currently popular QG candidates satisfy all three properties. There is of course an excellent reason for this: according to standard wisdom, the three properties in the main postulate are mutually incompatible.

Theorem 1 (No-go theorem, physical version) *There are no local observables in QG. In QFT, local observables are gauge-invariant unitary operators. Since diffeomorphisms are part of the gauge group of GR, any observable must be invariant under arbitrary diffeomorphisms, and hence it can not be local. The three properties of Postulate 1 are mutually exclusive.*

To gain some further insight, let us rephrase the postulate in terms of the representation theory of the diffeomorphism group.

Postulate 2 (Main postulate, representation theory version) *Quantum Gravity has the following properties:*

1. *All objects in the theory carry representations of the spacetime diffeomorphism group (the gravity property).*
2. *The representations are unitary and of lowest-energy type (the quantum property).*
3. *At least some representations are non-trivial (the field property).*

The no-go theorem can now be formulated as follows:

Theorem 2 (No-go theorem, representation theory version) *The space-time diffeomorphism group has no non-trivial, proper, unitary representations of lowest-energy type.*

This theorem is correct as stated, but no theorem is stronger than its axioms. The keyword is “proper”; if we relax that condition, the theorem no longer holds, as the following example illustrates.

Consider the group of diffeomorphisms on the circle, and its Lie algebra of vector fields $\mathfrak{vect}(S^1) \equiv \mathfrak{vect}(1)$; for brevity, the notation only

indicates the number of dimensions. The infinitesimal generators $L_m = -i \exp(imx) \partial / \partial x$, $m \in \mathbb{Z}$, satisfy

$$[L_m, L_n] = (n - m)L_{m+n}. \quad (1)$$

The only unitary lowest-energy representation of $\mathfrak{vect}(1)$ is the trivial one, in accordance with Theorem 2. However, it is well known from conformal field theory (CFT) how to solve this problem. $\mathfrak{vect}(1)$ admits a non-trivial central extension, the Virasoro algebra:

$$[L_m, L_n] = (n - m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n}, \quad (2)$$

where δ_m denotes the Kronecker delta and c is the central charge. A lowest-energy representation has a unique vacuum vector $|h\rangle$, which satisfies

$$\begin{aligned} L_0|h\rangle &= h|h\rangle, \\ L_{-m}|h\rangle &= 0, \quad \text{for all } -m < 0. \end{aligned} \quad (3)$$

The Virasoro algebra has non-trivial unitary representations of lowest-energy type, e.g. the entire Verma modules for $c > 1, h > 0$ or the discrete unitary series [3]:

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)}, \\ h &= \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \quad 1 \leq r < m, 1 \leq s \leq r. \end{aligned} \quad (4)$$

For these values of c and h , CFT satisfies all conditions in Postulate 2:

1. The theory has a symmetry under the diffeomorphism group on the circle.
2. The theory is unitary and the energy is bounded from below - the L_0 eigenvalue is at least h for every state in the Hilbert space.
3. The theory is local in the sense that correlation functions depend on separation. E.g., the correlator between two primary fields behaves like

$$\langle \phi(z)\phi(w) \rangle \approx (z - w)^{-2h} \quad (5)$$

when $z \rightarrow w$.

It is now clear how the no-go theorem can be avoided: allow projective representations of the spacetime diffeomorphism group.

Theorem 3 *To satisfy all desiderata in the main postulate it is necessary that symmetry of QG is some group extension of the spacetime diffeomorphism group. This converts the classical diffeomorphism gauge symmetry into an ordinary quantum symmetry, which does not need to commute with observables.*

On the Lie algebra level, this amounts to replacing $\mathfrak{vect}(d)$, the Lie algebra of vector fields in d dimensional spacetime, with a Lie algebra extension thereof. Since this extension generalizes the Virasoro algebra to multi-dimensional manifolds, we call it the multi-dimensional Virasoro algebra and denote it by $Vir(d)$; $Vir(1)$ is the ordinary Virasoro algebra.

2 The objections

Replacing $\mathfrak{vect}(d)$ with $Vir(d)$ is a drastic step, which may potentially lead to several objections.

1. $\mathfrak{vect}(d)$ does not possess any central extension at all when $d > 1$.
2. An extension of the diffeomorphism algebra is a diff anomaly. In QFT, there are no diff anomalies in four dimensions [1].
3. Diffeomorphisms are part of the gauge symmetries of gravity. In QFT observables are gauge-invariant operators, and hence all observables must commute with diffeomorphisms.
4. A diff anomaly is a kind of gauge anomaly, which automatically renders the theory inconsistent.

The first three objections are correct as formulated, but the statements contain assumptions that are overly strong. The last objection is manifestly false.

1. The diffeomorphism algebra in $d > 1$ dimensions does not possess any **central** extension, but it does possess non-central extensions that reduce to the Virasoro algebra in the case $d = 1$. $Vir(d)$ is an extension of $\mathfrak{vect}(d)$ by its module of closed $(d - 1)$ -forms. When $d = 1$, a closed zero-form is a constant function and the extension is central. When

$d > 1$, the extension does not commute with diffeomorphisms, but there are still non-trivial Lie algebra extensions.

The multi-dimensional Virasoro algebra is described explicitly in section 3. For a classification of abelian extensions of $\mathfrak{vect}(d)$ by modules of tensor fields, see [4].

2. There are no diff anomalies in four dimensions **within the framework of QFT**. However, the multi-dimensional Virasoro extensions described in section 3 certainly exist. Hence there are diff anomalies in arbitrary dimensions, in the same sense as the Virasoro central charge is a conformal anomaly in two dimensions, but these anomalies can not arise in QFT.

The off-shell representations of $\mathfrak{vect}(d)$ act on tensor fields and tensor densities. However, tensor densities are not a good starting point for quantization when $d > 1$; in higher dimensions, normal ordering gives rise to infinities coming from unrestricted sums over spatial degrees of freedom. Instead we must start from histories in the space of tensor-valued p -jets, p finite; locally, a p -jet is the same as a Taylor series truncated at order p . Since a p -jet history consists of finitely many functions of a single variable, normal ordering can be done without introducing any infinities.

A p -jet can be thought of as a regularization of the field, but not only so. A Taylor series does not only depend on the function being expanded, but also on the choice of expansion point, a.k.a. the observer's position. This is essential, because in all known representations of $Vir(d)$, the extension is a functional of the observer's trajectory. The Virasoro-like diff anomalies can not arise in QFT, because they depend on degrees of freedom not available. To construct these diff anomalies, we must replace QFT with a theory that depends on the observer's trajectory in addition to the fields. This theory is tentatively labelled Quantum Jet Theory (QJT).

The off-shell representations of $Vir(d)$ are explicitly described in section 4.

3. Diffeomorphisms generate a gauge symmetry **in the absense of diff anomalies**. A gauge anomaly converts a classical gauge symmetry into an ordinary quantum symmetry, which acts on the Hilbert space rather than reducing it. Hence there may be local observables in QG in the presence of diff anomalies.

4. It is simply not true that every theory with gauge anomalies is inconsistent. Counterexample: according to the no-ghost theorem, the free subcritical string can be quantized with a ghost-free spectrum despite its conformal gauge anomaly ([5], section 2.4). A gauge anomaly simply means that the classical and quantum theories have different symmetry groups.

This does of course not mean that every theory with a gauge anomaly can be rendered consistent, but the crucial consistency criterion is unitarity, not triviality. E.g., the gauge anomalies that appear in the standard model are related to the Mickelsson-Faddeev (MF) algebra¹ [13], which is known to lack good quantum representations; more precisely, the MF algebra has no non-trivial, unitary representations acting on a separable Hilbert space [14]. Gauge anomalies of this type must therefore cancel, which is also the case in the standard model. In contrast, $Vir(d)$ may well have non-trivial unitary representations (this is at least the case when $d = 1$), and such diff anomalies are not necessarily a sign of inconsistency.

Treating an anomalous gauge symmetry as a redundancy is of course inconsistent, since it becomes an ordinary symmetry after quantization.

3 Multi-dimensional Virasoro algebra

Denote by $Vir(d)$ the Virasoro algebra in d dimensions. In a Fourier basis on the d -torus, the generators are $L_\mu(m)$ and $S^\mu(m)$, $m = (m_0, m_1, \dots, m_{d-1}) \in \mathbb{Z}^d$, which satisfy

$$\begin{aligned}
[L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n) \\
&\quad + (c_1 m_\nu n_\mu + c_2 m_\mu n_\nu) m_\rho S^\rho(m+n), \\
[L_\mu(m), S^\nu(n)] &= n_\mu S^\nu(m+n) + \delta_\mu^\nu m_\rho S^\rho(m+n), \\
[S^\mu(m), S^\nu(n)] &= 0, \\
m_\mu S^\mu(m) &= 0.
\end{aligned} \tag{6}$$

To see that this algebra indeed reduces to the usual Virasoro algebra when $d = 1$, we notice that the condition $m_0 S^0(m_0) = 0$ implies that $S^0(m_0)$ is proportional to the Kronecker delta, which indeed commutes with diffeomorphisms. So the Virasoro extension is central when $d = 1$ but not otherwise.

¹ Note that the MF algebra is substantially different from the multi-dimensional affine algebra $Aff(d, \mathfrak{g})$ described in Section 6 below.

Nevertheless, (6) defines a well-defined and non-trivial Lie algebra extension of $\mathfrak{vect}(d)$ for every d .

The cocycle proportional to c_1 was discovered by Rao and Moody [16], and the one proportional to c_2 by myself [6]. We refer to c_1 and c_2 as *abelian charges*, in analogy with the central charge of $Vir(1)$.

$S^\mu(m)$ can be identified with the Fourier components of a $(d-1)$ -form:

$$\Omega(m) = \epsilon_{\mu_1 \mu_2 \dots \mu_d} S^{\mu_1}(m) dx^{\mu_2} \dots dx^{\mu_d}. \quad (7)$$

The last condition in (6) asserts that this $(d-1)$ -form is closed.

In the sequel we will use a different formulation not specific to tori. Let $\xi = \xi^\mu(x) \partial_\mu$ be a vector field, with commutator $[\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\nu \partial_\nu \xi^\mu \partial_\mu$. The Lie derivatives \mathcal{L}_ξ are the generators of $\mathfrak{vect}(d)$. $Vir(d)$ is defined by the following brackets

$$\begin{aligned} [\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \, \dot{q}^\rho(t) \left\{ c_1 \partial_\rho \partial_\nu \xi^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) + \right. \\ &\quad \left. + c_2 \partial_\rho \partial_\mu \xi^\mu(q(t)) \partial_\nu \eta^\nu(q(t)) \right\}, \\ [\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\ [q^\mu(t), q^\nu(t')] &= 0. \end{aligned} \quad (8)$$

The connection between (6) and (8) is given by

$$\begin{aligned} L_\mu(m) &= \mathcal{L}_{-i \exp(im \cdot x) \partial_\mu}, \\ S^\mu(m) &= \frac{1}{2\pi} \int dt \, \exp(im \cdot q(t)) \, \dot{q}^\mu(t). \end{aligned} \quad (9)$$

In particular, the closedness condition in (6) becomes $\int dt \, \frac{d}{dt} (\exp(im \cdot q(t))) \equiv 0$.

4 Off-shell representations

To construct Fock representations of $Vir(1)$ is straightforward:

- Start from classical fields, i.e. primary fields = scalar densities.
- Introduce canonical momenta.
- Normal order.

The first two steps of this procedure generalize nicely to higher dimensions, but the third step leads to infinites due to unrestricted sums over spatial directions. This is the reason why the representations of $Vir(d)$, $d \geq 2$, do not act on quantum fields.

Instead, we notice that $\mathfrak{vect}(d)$ can be embedded into a Heisenberg algebra with $2d$ generators q^μ and p_ν , and brackets

$$[q^\mu, p_\nu] = i\delta_\nu^\mu, \quad [q^\mu, q^\nu] = [p_\mu, p_\nu] = 0. \quad (10)$$

The embedding is given by

$$\mathcal{L}_\xi = i\xi^\mu(q)p_\mu. \quad (11)$$

Hence $\mathfrak{vect}(d)$ acts on the corresponding Fock module, which can be identified with the space of spacetime fields:

$$\mathcal{L}_\xi \Phi(q) = \xi^\mu(q) \partial_\mu \Phi(q). \quad (12)$$

Since the Heisenberg algebra (10) is finite-dimensional, the Fock representation of $\mathfrak{vect}(d)$ is proper. To obtain the extensions in (6), we need to find an embedding into an infinite-dimensional Heisenberg algebra. To this end, introduce infinitely many oscillators $q^\mu(t)$ and $p_\nu(t)$, $t \in S^1$, with non-zero brackets

$$[q^\mu(t), p_\nu(t')] = i\delta_\nu^\mu \delta(t - t'). \quad (13)$$

The embedding is given by

$$\mathcal{L}_\xi = i \int dt \xi^\mu(q(t)) p_\mu(t), \quad (14)$$

where the integral runs over $0 \leq t < 2\pi$.

Unlike the finite-dimensional case, the infinite-dimensional Heisenberg algebra (13) has several inequivalent Fock representations. To satisfy the quantum property, we must choose the one with energy bounded from below, where energy is identified with the frequency dual to the circle variable t . The Fock module consists of all functions of the positive-frequency Fourier components, plus half of the zero-frequency components.

However, the operators (14) do not act in a well-defined manner on this Fock space, because the action on the Fock vacuum is infinite. To remove this infinity, we must normal order. Because the oscillators $q^\mu(t)$ commute among themselves, this amounts to moving the positive-frequency components of $p_\mu(t)$ in (14) to the left. The normal ordered-operators satisfy

the multi-dimensional Virasoro algebra (8) with $c_1 = 2d$, $c_2 = 0$. $q^\mu(t)$ is the same in both (8) and (14).

More general Fock representations act on histories the the space of p -jets [8, 12], which locally can be identified with the space of Taylor series truncated at order p . Consider a spacetime field $\phi(x)$, expand it in a Taylor series around q^μ , and truncate at order p .

$$\phi(x) = \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \phi_{\mathbf{m}}(x - q)^{\mathbf{m}}, \quad (15)$$

where $\mathbf{m} = (m_0, m_1, \dots, m_{d-1})$, all $m_\mu \geq 0$, is a multi-index of length $|\mathbf{m}| = \sum_{\mu=0}^{d-1} m_\mu$, $\mathbf{m}! = m_0! m_1! \dots m_{d-1}!$, and

$$(x - q)^{\mathbf{m}} = (x^0 - q^0)^{m_0} (x^1 - q^1)^{m_1} \dots (x^{d-1} - q^{d-1})^{m_{d-1}}. \quad (16)$$

The space of p -jets is spanned by the Taylor coefficients $\phi_{\mathbf{m}}$, $|\mathbf{m}| \leq p$ and the expansion point q^μ .

Now consider p -jet histories by letting everything depend on an extra circle parameter $t \in S^1$. The Heisenberg algebra is spanned by the oscillators $q^\mu(t)$, $p_\nu(t)$, $\phi_{\mathbf{m}}(t)$, and $\pi^{\mathbf{n}}(t)$, obeying (13) and

$$[\phi_{\mathbf{m}}(t), \pi^{\mathbf{n}}(t')] = i \delta_{\mathbf{m}}^{\mathbf{n}} \delta(t - t'). \quad (17)$$

After normal ordering, denoted by double dots $: \cdot :$, we obtain a projective Fock representation of the diffeomorphism algebra

$$\mathcal{L}_\xi = i \int dt \left\{ : \xi^\mu(q(t)) p_\mu(t) : - \sum_{\mathbf{m}, \mathbf{n}} : \pi^{\mathbf{n}}(t) T_{\mathbf{n}}^{\mathbf{m}}(\xi(q(t))) \phi_{\mathbf{m}}(t) : \right\}, \quad (18)$$

where the sum runs over all \mathbf{m} and \mathbf{n} such that $|\mathbf{m}| \leq |\mathbf{n}| \leq p$. $T_{\mathbf{n}}^{\mathbf{m}}(\xi)$ are some functions of ξ^μ and its derivatives up to order $p+1$, explicitly written down in [8].

The construction is readily generalized to fermionic fields, but the expansion point q^μ is of course always bosonic.

A major shortcoming of this construction is that only linear representations have been considered. In physics, we are ultimately interested in unitary representations, but we have nothing to say about that.

The jet data have a natural physical interpretation. The expansion point q^μ is the observer's position and p_μ his momentum. The Taylor coefficients $\phi_{\mathbf{m}}$ are the excitations of the field that the observer can measure with a local detector.

5 Reparametrizations and the messy cocycles

By passing to histories in the space of p -jets, the Taylor series (15) is replaced by a field that depends both on the spacetime coordinates x^μ and the trajectory parameter t :

$$\phi(x, t) = \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \phi_{\mathbf{m}}(t) (x - q(t))^{\mathbf{m}}. \quad (19)$$

The natural algebra that acts on such a field is $\mathfrak{vect}(d) \oplus \mathfrak{vect}(1)$, where the first factor describes spacetime diffeomorphisms and the second reparametrizations of the observer's trajectory. We can thus enlarge the representations in the previous section to this larger algebra, by embedding the extra $\mathfrak{vect}(1)$ generators into the same Heisenberg algebra.

Upon normal ordering, $\mathfrak{vect}(d) \oplus \mathfrak{vect}(1)$ acquires four different cocycles and becomes $Vir(d) \hat{\oplus} Vir(1)$. $Vir(d)$ has two cocycles, $Vir(1)$ has one, and the final cocycle is found in the cross term, which is indicated by the notation $\hat{\oplus}$. Explicit formulas can be found in [8].

Whereas this is the mathematically clean formulation, it is somewhat redundant from a physical point of view. There are two different time coordinates: the x^0 coordinate and the parameter t . To bring out the physical content, we can use a trick described in [8]. Before normal ordering, we may pretend that all brackets are Poisson brackets. To eliminate reparametrizations, we impose the constraint that all $\mathfrak{vect}(1)$ generators vanish:

$$L(t) \approx 0. \quad (20)$$

Supplement this first-class constraint with a gauge condition to make it second class:

$$q^0(t) \approx t. \quad (21)$$

Finally, the constraint is eliminated by replacing Poisson bracket by Dirac brackets. It turns out that the diffeomorphism generators \mathcal{L}_ξ still satisfy an extension of $\mathfrak{vect}(d)^2$ by four different cocycles.

The number of cocycles must be exactly four, because this is how many cocycles $\mathfrak{vect}(d) \oplus \mathfrak{vect}(1)$ has. Apart from the two cocycles in (8), there are two very complicated cocycles, which are anisotropic in the sense that the dependence on the zeroth coordinate is different from the others. They are

² That the regular terms still generate $\mathfrak{vect}(d)$ is non-trivial and can be seen by an explicit calculation.

colloquially known as the *messy cocycles*. The anisotropy clearly comes from the gauge condition (21). For an explicit description of the messy cocycles, see [7].

If we now also set $x^0 = t$, the field (19) becomes independent of all Taylor coefficients $\phi_{\mathbf{m}}(t)$ with $m_0 > 0$, because that term is proportional to

$$(x^0 - q^0(t))^{m_0} = (t - t)^{m_0} = 0. \quad (22)$$

We thus have eliminated the x^0 coordinate altogether, and constructed a field $\phi(\mathbf{x}, t)$ which depends on a single time coordinate t and the spatial coordinates \mathbf{x} . The price is that the projective action of $\mathbf{vect}(d)$ on the Fock space is very complicated.

6 Multi-dimensional affine algebra

There is an analogous multi-dimensional affine algebra ([15] section 4). Let $\mathbf{map}(d, \mathfrak{g})$ be the algebra of maps from d -dimensional space to a Lie algebra \mathfrak{g} with basis J^a , structure constants f^{abc} , and Killing metric δ^{ab} . $Aff(d, \mathfrak{g})$ is defined by the brackets

$$[\mathcal{J}_X, \mathcal{J}_Y] = \mathcal{J}_{[X, Y]} - \frac{k}{2\pi i} \delta^{ab} \int dt \dot{q}^\rho(t) \partial_\rho X_a(q(t)) Y_b(q(t)), \quad (23)$$

where $X = X_a(x) J^a$ is a \mathfrak{g} -valued function. In the Fourier basis, this becomes

$$[J^a(m), J^b(n)] = i f^{abc} J^c(m+n) - k \delta^{ab} m_\mu S^\mu(m+n). \quad (24)$$

The $Aff(d, \mathfrak{g})$ generators commute with $q^\mu(t)$ and $S^\mu(m)$, and admit an intertwining action of $Vir(d)$.

Note that this cocycle is proportional to the second Casimir operator. $Aff(d, \mathfrak{g})$ is thus unrelated to the gauge anomalies appearing in the standard model, which are proportional to the third Casimir.

Off-shell representations of $Aff(d, \mathfrak{g})$ are constructed in analogy with $Vir(d)$. Let M^a denote matrices in some finite-dimensional representation of \mathfrak{g} . The following expression defines an embedding of $Aff(d, \mathfrak{g})$ into the Heisenberg algebra:

$$\mathcal{J}_X = -i \int dt \sum_{\mathbf{m}, \mathbf{n}} : \pi^{\mathbf{n}}(t) J_{\mathbf{n}}^{\mathbf{m}}(X(q(t))) \phi_{\mathbf{m}}(t) :, \quad (25)$$

where

$$J_{\mathbf{n}}^{\mathbf{m}}(X) = \binom{\mathbf{n}}{\mathbf{m}} \partial_{\mathbf{n}-\mathbf{m}} X_a M^a. \quad (26)$$

Hence $Aff(d, \mathfrak{g})$ acts on the Fock space.

Whereas the off-shell representations of $Vir(d)$ are only understood at the linear level, unitary representations of $Aff(d, \mathfrak{g})$ are easily constructed. Specialize (25) to zero-jets:

$$\mathcal{J}_X = \int dt X_a(q(t)) J^a(t), \quad (27)$$

where

$$J^a(t) = -i : \pi^0(t) M^a \phi_0(t) :. \quad (28)$$

To verify that (27) satisfies $Aff(d, \mathfrak{g})$, it suffices to prove that the operators $J^a(t)$ satisfy the affine algebra $Aff(1, \mathfrak{g}) = \widehat{\mathfrak{g}}$. Conversely, we obtain an $Aff(d, \mathfrak{g})$ representation for every $\widehat{\mathfrak{g}}$ representation. If this representation is unitary, so is the $Aff(d, \mathfrak{g})$ representation (27), since \mathcal{J}_X is merely a linear combination of unitary operators.

Note that we could replace $q^\mu(t)$ with a c-number because $[\mathcal{J}_X, q^\mu(t)] = 0$. This is not possible when diffeomorphisms are taken into account, so a similar trick is not possible for $Vir(d)$.

The representation (27) should be equivalent to a representation induced from $\widehat{\mathfrak{g}}$, living on the loop $q^\mu(t)$. As such, it is presumably covered by the discussion in [15], Section 9.1. Pressley and Segal found this result rather disappointing, but the reason why the irreps only require zero-jets is that the current algebra does not explore neighboring spacetime points. The circle $q^\mu(t)$ commutes with everything in sight and can therefore be replaced with a c-number; the extension becomes central. In physics, there is always an intertwining action of diffeomorphisms or some subgroup thereof, such as the Poincaré group. Once such spacetime groups are taken into consideration, $q^\mu(t)$ becomes an operator, the extension is no longer central, and interesting representations depend on more than $\widehat{\mathfrak{g}}$.

7 Conclusion

Locality is compatible with diffeomorphism symmetry, but only in the presence of diff anomalies. It is necessary to quantize histories in the space of

p -jets, rather than quantizing the fields themselves. The appropriate name for this quantum theory is Quantum Jet Theory (QJT).

Some attempts to formulate the physical consequences of QJT have been made [10, 11], but the results were inconclusive. Nevertheless, we can make some general observations based on the structure of the off-shell representations.

- Since diff anomalies of the type in (6) can not arise in QFT, QJT is substantially different from QFT.
- Passage to p -jets is a kind regularization, because a problem in QFT is replaced by a problem with fewer degrees of freedom.
- We ultimately want to remove the regulator, which amounts to taking the jet order $p \rightarrow \infty$. The abelian charges diverge in this limit. It is not surprising that the field theory problems resurface in the field theory limit.
- The abelian charges are polynomials in p of order d , where d is the dimension of spacetime. The leading terms can be cancelled in representations acting on several fields, both bosonic and fermionic, making the total abelian charges finite when $p \rightarrow \infty$ [9].
- There are intriguing hints that cancellation of the leading terms in the abelian charges works best in four dimensions [10, 11].
- QJT is the unique regularization that preserves diffeomorphism symmetry exactly.
- The anomalies arise because QJT does not preserve the dynamics. If the equations of motion have order n , only Taylor coefficients of order up to $p - n$ have dynamics. The rest have equations of motion which involve Taylor coefficients of order higher than p , and thus lie outside p -jet space.
- QJT is more than a regularization of QFT, because a Taylor series depends not only on the field being expanded, but also on the expansion point q^μ . This is naturally identified as the observer's position in spacetime.
- There are two types of observables in QJT: the Taylor coefficients, which are field excitations confined to a local neighborhood of the observer, and the observer's position. Both are operators, measured

by some detectors and subject to quantum fluctuations. To obtain the field at a fixed point, both the field inside the detector and the detector's location are needed. The second measurement introduces an extra fuzziness not present in QFT.

- The uncertainty in the observer's position is eliminated if the observer's inert mass M diverges, since the bracket between the observer's position and velocity is proportional to \hbar/M (non-relativistically). In this limit QJT should be physically equivalent to QFT.
- Making the observer's inert mass infinite leads to problems with gravity, since inert mass equals heavy mass. In my opinion, this is the physical reason why QFT and GR are incompatible; they tacitly make incompatible assumptions about the observers mass:
 - In GR, the observer's heavy mass is assumed to vanish, so the observer does not disturb the fields.
 - In QFT, the observer's inert mass is assumed to be infinite, so the fields do not disturb the observer.
- The abelian charges are generalizations of the Virasoro central charge, which is known to couple to length scales. E.g. in Einsteins gravity in three-dimensional AdS space, the central charge $c = -3\ell/2G$, where ℓ is the AdS radius and G is Newtons constant [2]. By analogy, we expect that the abelian charges are manifested in the large-scale structure of the universe, e.g. as a cosmological constant.
- If QG has local observables, any result that assumes that QG is a non-local theory is questionable.

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